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Packet-Based Model Predictive Control for Networked Control Systems With Random Packet Losses

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Abstract— **In this paper, the stability for a class of nonlinear networked control systems with a model predictive controller (MPC) is investigated. Both the sensor-to-controller channel and the controller-to-actuator channel suffer from random packet losses. By constructing a novel cost function, and studying its deviation from the original MPC cost function, we establish the stochastic stability for the closed-loop system. To guarantee the stability, the relationship between the prediction horizon and the packet loss probabilities of two channels is also discussed. Finally, the effectiveness of our results is demonstrated by a numerical example.**

I. Introduction

Networked control system (NCS) is a class of control system where the controller communicates with sensors and actuators through digital network [1], [2], [3]. NCS has recently attracted the attention of a lot of researchers due to its merits such as increased flexibility, reduced wiring and lower maintenance costs, etc. However, since the wireless communication medium is shared and inherently unreliable, numerous challenges are also brought to the analysis and design of control system. Packet loss is a typical detrimental phenomenon that may result from the channel fading and packet collision, and will deteriorate the control performance or even cause instability [4].

Various methods have been proposed to deal with packet loss, among which packet-based predictive control method is promising since it can actively compensate the packet losses and then obtain better control performance, see, e.g. [5], [6], [7], [8]. Specifically, the remote controller calculates a sequence of predicted control signals, lumps them into one packet and sends the packet to the actuator. The actuator will then choose a proper input according to whether the packet is dropped or not [5].

A key problem that how to calculate the predicted control signal arises. Model predictive control (MPC) is one of popular prediction techniques has been widely adopted in chemical process and mechanical systems, due to its ability on handling constraints and optimizing closed-loop performance systematically [9], [10], [11]. Through solving an optimization problem at each time step, a sequence of control signals that results in improved control performance can be obtained. Inspired by these facts, a packet-based MPC framework is proposed to deal with NCS with communication constraints. In [12], the authors study a predictive control formulation for discrete-time nonlinear systems with random packet losses. This work is generalized to the nonlinear systems with bounded disturbances case, where the packet losses is modeled as Markov chain in [13] and the number of consecutive packet losses is assumed bounded in [14]. Lješnjanin *et al.* [15] considers a situation where only one plant input node can access the network at each time instant, and proposes a MPC-based scheduling strategy to mitigate this limitation and proves the stability. [16] studies the random channel access mechanism for the sensors of multiple subsystems communicating competitively over shared channel with remote controller, and verifies that the stability of each subsystem is achievable. All of these works are in the context of one-channel packet losses. The NCS with two-channel packet losses is more general, and the stability analysis for such system is much more challenging. Since the exact state of the plant is not available to the model predictive controller at each time instant, the analysis method proposed in [13] can not be directly applied.

With the above inspiration, we perform the stability analysis for discrete-time nonlinear system with packet-based MPC in the presence of two-channel random packet losses. The main contributions of this paper are summarized as follows.

1) The stochastic Lyapunov function constructed in our work depends on the exact state and the control sequence calculated by the estimated state, which is a critical difference from [13], where the optimal MPC cost function serves as the Lyapunov function. Based on the decreasing property of the constructed Lyapunov function at the successful transmission instants of actuator, the conditions for stochastic stability are then developed.

2) To guarantee the stochastic stability, the relation of the prediction horizon and the packet loss probabilities of sensor-to-controller (S-C) channel and controller-to-actuator (C-A) channel is investigated. The relation proposed in [13] and [16] can be viewed as a special case of ours.

3) In a similar work [17], the authors proved that the closed-loop system is input-to-state practical stability, and the ultimate bound will not converge to zero even for the disturbance-free system. In contrast, the stochastic stability condition is established in this paper and the ultimate bound

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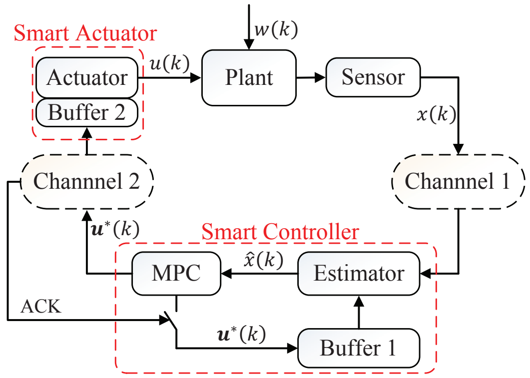

Fig. 1. The architecture of NCS

depends only on the boundary of the disturbance.

The rest of the paper is organized as follows. The structure of the NCS and some preliminaries are introduced in Section II. In Section III, we present some necessary notions and assumptions, and also give some useful preliminary results. In Section IV, the stochastic stability of the closed-loop system is established. A numerical example is shown in Section V to illustrate the effectiveness of our results. Finally, Section VI concludes the paper.

Notations. Throughout this paper, $\mathbb{R}^n$ represents the n-dimensional Euclidean space, $\mathbb{N}_0 \triangleq \{0, 1, 2, \dots\}$. 0_n and I_n stand for the $n \times n$-dimensional zero matrix and identity matrix, respectively. For a vector x, x^T represents the transpose of x and $\|x\|$ means the Euclidean norm of x. Sequence $\{y(k)\}_{k\in\mathbb{N}_0}$ and sequence $\{w\}_{l_1}^{l_2}$ represent $\{y(0), y(1), \dots\}$ and $\{w(l_1), w(l_1+1), \dots w(l_2)\}$, respectively. We use $\Pr\{\mathcal{E}\}$ to denote the probability of an event $\mathcal{E}$, and $\Pr\{\mathcal{E}_1 | \mathcal{E}_2\}$ to denote the conditional probability of $\mathcal{E}_1$ when given $\mathcal{E}_2$. $E\{v\}$ denotes the expectation of random variable v, and $E\{v|\mathcal{E}\}$ is the conditional expectation of v when given $\mathcal{E}$.

II. Problem Formulation

The plant is described by the following discrete-time nonlinear system

$$x(k+1) = f(x(k), u(k), w(k)), \quad k \in \mathbb{N}_0 \qquad (1)$$

where $f(0,0,0) = 0$. $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{U} \subset \mathbb{R}^p$ is control input and compact set $\mathbb{U}$ is the control constraint set. The disturbance $\{w(k)\}_{k\in\mathbb{N}_0}$ is an i.i.d random variable sequence with arbitrary distributions and satisfies $E\{\|w(k)\|^s\} < \infty$. The initial state $x(t_{ini}) = x_0$ is also arbitrarily distributed and satisfies $E\{\|x_0\|^s\} < \infty$.

It is convenient to introduce the following iterated mappings:

$$f^i(x(k), \{w\}_0^{i-1}) \triangleq f(f^{i-1}(x(k), \{w\}_0^{i-2}), u_{i-1}, w(i-1))$$
$$\bar{f}^i(x(k)) \triangleq f(\bar{f}^{i-1}(x(k)), u_{i-1}, 0)$$

for $i \in \mathbb{N}_0$, and where $f^0((x(k), \{w\}_0^{-1}) = \bar{f}^0((x(k)) = x(k)$, $\{w\}_0^{-1} = \{\}$.

We first give an outline of the configuration of NCS. The configuration of NCS illustrated in Fig.1 is very similar to that in [17], which consists of nonlinear plant, sensor, smart controller, smart actuator and communication network. Supposed that the state and control information is transmitted through data network, where the size of the data packet frame can be very large. Therefore, the predicted control sequence calculated by the controller can be encapsulated into one packet and then send it to the actuator. For the S-C channel, both TCP-like protocol and UDP-like protocol can be adopted. In contrast, the TCP-like protocol is adopted for the C-A channel. Unlike the UDP-like protocol, an acknowledgment packet will be sent to the controller once the control sequence is received by the smart actuator. Thanks to the acknowledgment packet, the actual inputs of the plant are kept available to the smart controller. It is assumed that the acknowledgment packet is transmitted without any packet dropouts and time delays. The smart controller contains three parts: state estimator, MPC and buffer 1. Buffer 1 stores the control sequence and provides the actual control input for the estimator based on the acknowledgement signal. The state estimator gives an estimation of the actual state based on the system model (1) and actual input provided by buffer 1 especially when the packet loss of S-C channel occurs. MPC calculates the control sequence based on the estimated state by solving an constrained optimization problem. The smart actuator is composed of an actuator and buffer 2 which is also used to store the received control sequence and provides the control signal to actuator. The detailed introduction for each component can be seen in the following subsections.

A. Packet Loss Model

Due to the random access mechanism and the transmission errors, the packet losses are inevitable. We model the communication channel as erasure channels, i.e., the models of channel 1 and channel 2 depicted in Fig.1 are characterized by following two discrete Bernoulli processes $\{d_s(k)\}_{k\in\mathbb{N}_0}$ and $\{d_a(k)\}_{k\in\mathbb{N}_0}$, respectively.

$$d_s(k) = \begin{cases} 0 & \text{if the controller receives the state packet} \\ 1 & \text{if the packet dropout of channel 1 occurs} \end{cases}$$

$$d_a(k) = \begin{cases} 0 & \text{if the actuator receives the control packet} \\ 1 & \text{if the packet dropout of channel 2 occurs} \end{cases}$$

Each variable $d_s(k)$ or $d_a(k)$ is i.i.d with packet loss probabilities

$$\Pr\{d_s(k) = 1\} = p_s, \;\; \Pr\{d_s(k) = 0\} = 1 - p_s,$$
$$\Pr\{d_a(k) = 1\} = p_a, \;\; \Pr\{d_a(k) = 0\} = 1 - p_a.$$

Fig. 2 provides an illustration to describe the two-channel packet losses. Denote t_k as the $(k+1)$-th time instant that actuator receives the control sequence, and denote t_k^- as the time when the state packet is the last time received by the controller *before or at* t_k. t_k^+ represents the time when the state packet is first received by the controller *after* t_k (i.e. $t_k^+ \geq t_k + 1$). The sensor sends packet to remote controller at each time step. However, the controller does not calculate and send the control sequence during $\{t_k + 1, \dots, t_k^+ - 1\}$ with the reason shown in Remark 3.

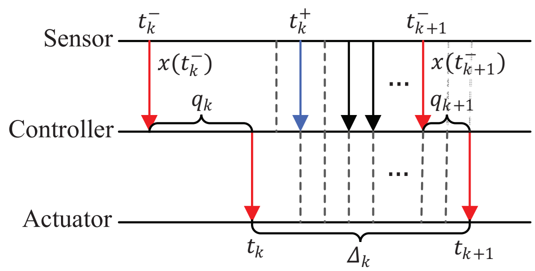

Fig. 2. Two-channel packet losses

Here, we suppose that the initial time $t_{ini} = 0$ is the first time that the controller receives the state information. One interpretation can be that the controller knows the initial state. After that, the controller must calculate the control sequence and send it to the actuator at each time step until the actuator successfully receives the control packet at time t_0. So we have the following probability

$$\Pr\{t_0 - t_{ini} = d\} = p_a^d(1-p_a) \tag{2}$$

It should be further emphasized that the actual control inputs of the plant are 0 during time interval $[t_{ini}, t_0 - 1]$.

Remark 1: In [17], the numbers of consecutive packet losses of S-C channel and C-A channel are assumed to be bounded. In contrast, the two-channel packet losses are characterized in this paper by two Bernoulli processes, which is especially fit for modeling the random access communication strategies in scheduling issues, see, e.g. [16][18], and consequently, the maximum number of consecutive dropouts of both channel can be unbounded.

B. Buffering

Two buffers are deployed at remote controller side and actuator side, respectively. Both two buffers store the received control sequence. Since the TCP-like protocol is adopted for the C-A communication network, the acknowledgement signal sent to controller informs whether the actuator has received the packet or not. Thanks to this acknowledgement signal, the contents of both buffers can always keep consistent, i.e. if the packet loss occurs, the contents of both buffers shift; otherwise, the contents are overwritten by new received packet. More formally,

$$\begin{aligned} \mathbf{b}(k) &= d_a(k)S\mathbf{b}(k-1) + (1-d_a(k))\mathbf{u}(k) \\ u(k) &= e_1^T\mathbf{b}(k) \end{aligned} \tag{3}$$

where $\mathbf{b}(k)$ is the contents of both buffers with $\mathbf{b}(0) = 0$ and $\mathbf{b}(t_0^-) = 0$, $\mathbf{u}(k)$ is the control sequence and $u(k)$ is the actual control input. S and e_1 are defined via:

$$S \triangleq \begin{bmatrix} 0_p & I_p & 0_p & \dots & 0_p \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_p & \dots & 0_p & I_p & 0_p \\ 0_p & \dots & \dots & 0_p & I_p \\ 0_p & \dots & \dots & \dots & 0_p \end{bmatrix}, \quad e_1 \triangleq \begin{bmatrix} I_p \\ 0_p \\ \vdots \\ 0_p \end{bmatrix}$$

C. Smart Controller

As previously mentioned, smart controller contains three parts. For the estimator, if the state packet is received successfully, then the estimated state is the actual state; otherwise, the estimated state can be calculated by the last time estimation and the actual input. That is,

$$\hat{x}(k+1) = \begin{cases} x(k+1) & \text{if } d_s(k) = 0 \\ f(\hat{x}(k), u(k), 0) & \text{if } d_s(k) = 1 \end{cases} \tag{4}$$

where $u(k)$ is defined in (3).

For the MPC part, the constrained optimization problem is solved and control sequence is generated according to the estimated state. Specifically, the optimization problem is formulated as:

Problem 1:

$$\min V_N(\hat{x}(k), \mathbf{u}(k))$$

subject to

$$\begin{aligned} &\hat{x}_{i+1}(k) = f(\hat{x}_i(k), u_i(k), 0) \\ &\hat{x}_0(k) = \hat{x}(k) \\ &u_i(k) \in \mathbb{U}, \quad \forall i = 0, \dots, N-1 \end{aligned}$$

where $\hat{x}(k)$ is the estimated state, and $V_N(\hat{x}(k), \mathbf{u}(k)) = \sum_{i=0}^{N-1} l(\hat{x}_i(k), u_i(k)) + V_f(\hat{x}_N(k))$ is the MPC cost function. $l(.)$ and $V_f(.)$ are the stage cost and the terminal cost, respectively. Notice that the estimated state at time t_k is of great importance because the obtained control sequence is received by the actuator, so we denote it as $\hat{x}(t_k|t_k^-)$, which can be calculated by $\hat{x}(t_k|t_k^-) = \bar{f}^{q_k}(x(t_k^-))$.

Let $\mathbf{u}^*(t_k) = \arg\min V_N(\hat{x}(t_k|t_k^-), \mathbf{u}(t_k))$ represent the optimal control sequence of the above optimization problem at time t_k, and is denoted by $\{u_0^*(t_k), \dots, u_{N-1}^*(t_k)\}$. Based on this optimal control sequence, the corresponding state sequence is $\{\hat{x}(t_k|t_k^-), \hat{x}_1^*(t_k|t_k^-), \dots, \hat{x}_N^*(t_k|t_k^-)\}$. The optimal cost-to-go is denoted by $V_N^*(\hat{x}(t_k|t_k^-))$. The obtained optimal control sequence is merged into one packet and then transmitted to the actuator.

Remark 2: In contrast to conventional MPC algorithm, the initial value here is the estimated state rather than the exact state [10], [13], [16]. If $t_k - t_k^- = 0$, that is, the control packet is not dropped at time t_k, The above MPC algorithm is then reduced to the conventional one.

Remark 3: Problem 1 will not be solved for all time instants. This MPC algorithm is carried out if further information is provided. That is to say, the MPC does not need to take any action during the time interval (t_k, t_k^+) since no new state information is received by the controller and the latest control sequence has been received by the actuator. By this way, the computation load of the controller and the communication load of the C-A channel can be reduced. For a more detailed explanation of this MPC algorithm we refer to [17].

III. Some Preliminaries

In this section, we give some necessary assumptions and lemmas, which are crucial to the following procedures.

Assumption 1: [13] The plant model and the cost functions are all uniformly continuous, i.e., there exist constants $\lambda_x, \lambda_w, \lambda_l, \lambda_f$ such that for all $(x, y, u, w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^m$,

$$\|f(x, u, w) - f(y, u, 0)\|^s \le \lambda_x \|x - y\|^s + \lambda_w \|w\|^s \quad (5)$$

$$|l(x, u) - l(y, u)| \le \lambda_l \|x - y\|^s \quad (6)$$

$$|V_f(x) - V_f(y)| \le \lambda_f \|x - y\|^s \quad (7)$$

Assumption 2: [13] The stage cost $l(.)$ and terminal cost $V_f(.)$ satisfy, for all $(x, u) \in \mathbb{R}^n \times \mathbb{U}$,

$$l(x, u) \ge \alpha_l \|x\|^s \quad (8)$$

$$V_f(x) \ge \alpha_f \|x\|^s \quad (9)$$

where α_l and α_f are two positive constants.

Assumption 3: [13] There exists a constrained control law $\kappa : \mathbb{R}^n \to \mathbb{U}$ such that

$$V_f(f(x, \kappa(x), 0)) + l(x, \kappa(x)) \le V_f(x) \quad (10)$$

for all $x \in \mathbb{R}^n$.

According to Assumption 3, the following result can be easily obtained.

Lemma 1: [12] If Assumption 3 holds, then

$$l(\hat{x}(t_k|t_k^-), u_0^*(t_k)) \le V_N^*(\hat{x}(t_k|t_k^-)) \le V_f(\hat{x}(t_k|t_k^-)) \quad (11)$$

holds for all $\hat{x}(t_k|t_k^-) \in \mathbb{R}^n$.

Assumption 4: For the open-loop system, there exist constant γ which satisfies $p_a\gamma < 1$ and $p_s\gamma < 1$, and η such that

$$V_f(f(x, 0, w)) \le \gamma V_f(x) + \eta \|w\|^s \quad (12)$$

for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$.

Similar to Lemma 5 in [13], the above assumption is satisfied with $\gamma = \lambda_f \lambda_x / \alpha_f$ and $\eta = \lambda_f \lambda_x$ if $\max\{p_s, p_a\}\lambda_f \lambda_x / \alpha_f < 1$. Besides, Assumption 4 implies that $V_f(f(x, 0, 0)) \le \gamma V_f(x)$ holds for the nominal open-loop system.

The following conditional probabilities are essential in deriving the final results.

Lemma 2: For simplicity, we write $\mathcal{E}_1^k \triangleq \{t_k - t_k^- = q_k\}$, $\mathcal{E}_2^k \triangleq \{t_{k+1} - t_k = \Delta_k\}$ and $\mathcal{E}_1^{k+1} \triangleq \{t_{k+1} - t_{k+1}^- = q_{k+1}\}$, where $q_k \ge 0$, $\Delta_k > q_{k+1} \ge 0$, then we have

$$\Pr\{\mathcal{E}_2^k, \mathcal{E}_1^{k+1} | \mathcal{E}_1^k\} = \frac{(1 - p_s)(1 - p_a)}{p_a - p_s} \Big[(1 - p_s) p_s^{q_{k+1}} p_a^{\Delta_k} - (1 - p_a) p_a^{q_{k+1}} p_s^{\Delta_k} \Big] \quad (13)$$

$$\Pr\{\mathcal{E}_2^k | \mathcal{E}_1^k\} = \frac{(1 - p_s)(1 - p_a)(p_a^{\Delta_k} - p_s^{\Delta_k})}{p_a - p_s} \quad (14)$$

for $p_a \ne p_s$,

$$\Pr\{\mathcal{E}_2^k, \mathcal{E}_1^{k+1} | \mathcal{E}_1^k\} = (1 - p_s)^2 p_s^{\Delta_k + q_{k+1} - 1} \big[p_s + (\Delta_k - q_{k+1})(1 - p_s) \big] \quad (15)$$

$$\Pr\{\mathcal{E}_2^k | \mathcal{E}_1^k\} = (1 - p_s)^2 \Delta_k p_s^{\Delta_k - 1} \quad (16)$$

for $p_a = p_s$, and

$$\Pr\{\mathcal{E}_1^{k+1} | \mathcal{E}_1^k\} = (p_s p_a)^{q_{k+1}} - (p_s p_a)^{q_{k+1}+1}. \quad (17)$$

Proof: For $\Pr\{\mathcal{E}_2^k, \mathcal{E}_1^{k+1} | \mathcal{E}_1^k\}$, one can list all possible cases and sum the probability of each case. For simplicity, denoting q_{k+1} and Δ_k by q and Δ, respectively, we have

$$\begin{aligned}
&\Pr\{\mathcal{E}_2^k, \mathcal{E}_1^{k+1} | \mathcal{E}_1^k\} \\
=&(1 - p_s)^2 p_s^q p_a^{\Delta - 1}(1 - p_a) + (1 - p_s)^2 p_s^{q+1} p_a^{\Delta - 2}(1 - p_a) \\
&+ \cdots + (1 - p_s)^2 p_s^{\Delta - 2} p_a^{q+1}(1 - p_a) \\
&+ (1 - p_s) p_s^{\Delta - 1} p_a^q (1 - p_a)
\end{aligned}$$

If $p_a = p_s$, rearranging the above equation yields (15). If $p_a \ne p_s$, by the sum of geometrical sequence, we get (13).

$\Pr\{\mathcal{E}_2^k | \mathcal{E}_1^k\}$ and $\Pr\{\mathcal{E}_1^{k+1} | \mathcal{E}_1^k\}$ are the marginal probability distributions of $\Pr\{\mathcal{E}_2^k, \mathcal{E}_1^{k+1} | \mathcal{E}_1^k\}$, and can be easily calculated. ■

When $q_{k+1} = 0$ and $p_s = 0$ (or $p_a = 0$), some undefined terms appear. In our problem setups, these undefined terms (e.g. $p_a^{q_{k+1}}$, $p_s^{q_{k+1}}$, $(p_s p_a)^{q_{k+1}}$) can be simply set to 1. It should be indicated that the right-hand sides of the above equations do not contain q_k. So event $\mathcal{E}_1^{k+1}$ and event $\mathcal{E}_2^k$ are both independent of event $\mathcal{E}_1^k$.

Remark 4: The limit of the result for $p_a \ne p_s$ as p_a approaches p_s equals to the corresponding result for $p_a \ne p_s$. For example, taking limit for (13) and using L'Hospitol principle gives Equation (15). Equation (16) can be obtained by the same way. Therefore, we use only (13) and (15) in the following procedures for convenient.

It is worth pointing out that both the sequence $\{q_k\}_{k \in \mathbb{N}_0}$ and the sequence $\{\Delta_k\}_{k \in \mathbb{N}_0}$ are i.i.d., respectively. Unfortunately, the state sequences $\{x(t_k^-)\}_{k \in \mathbb{N}_0}$ and $\{\hat{x}(t_k|t_k^-)\}_{k \in \mathbb{N}_0}$ are in general not Markovian. To handle this problem, an extended state $\xi(t_k) = [x(t_k^-)^T, \mathbf{b}(t_k^-)^T]^T$ is constructed, where $\mathbf{b}(t_k^-)$ defined in (3) is the content of buffer 1 at time t_k^-. It can be verified that the sequence $\{\xi(t_k)\}_{k \in \mathbb{N}_0}$ is Markovian, i.e. $\xi(t_{k+1})$ depends solely on $\xi(t_k)$ and is independently from the history.

IV. Stability Analysis

For the closed-loop system, we establish the stability by the following constructed cost function:

$$J_N(x(t_k)) = \sum_{i=0}^{N-1} l(\bar{f}^i(x(t_k)), u_i^*(t_k)) + V_f(\bar{f}^N(x(t_k)))$$

Noting that $V_N^*(\hat{x}(t_k|t_k^-))$ is the optimal cost of the MPC problem which based on the estimated state $\hat{x}(t_k|t_k^-)$, while $J_N(x(t_k))$ is dependent on the actual state $x(t_k)$ and the optimal control sequence $\mathbf{u}^*(t_k)$ of *Problem* 1.

The following technical lemma is often used in the proof below.

Lemma 3: If Assumption 1 holds, then we have

$$\begin{aligned}
&E\big\{\|\hat{x}_i(t_{k+1}|t_{k+1}^-) - \hat{x}_{i+\Delta_k}^*(t_k|t_k^-)\|^s \\
&\qquad\qquad \big|\xi(t_0), \mathcal{E}_1^k, \mathcal{E}_2^k, \mathcal{E}_1^{k+1}\big\} \\
\le&\beta_{k,i} E\|w\|^s \triangleq \lambda_w \frac{\lambda_x^{i+q_{k+1}} - \lambda_x^{i+\Delta_k+q_k}}{1 - \lambda_x} E\|w\|^s
\end{aligned}$$

for all $i \in \{0, 1, \ldots, N - \Delta_k\}$, with $\Delta_k \le N$.

Due to the limited space, the proof is omitted here.

We start the proof procedure with $k=0$. Denotes the optimal control sequence at time t_0 by $\{u_0^*(t_0),\ldots,u_{N-1}^*(t_0)\}$ and the corresponding state sequence based on estimated state is $\{\hat{x}(t_0|t_0^-),\hat{x}_1^*(t_0|t_0^-),\ldots,\hat{x}_N^*(t_0|t_0^-)\}$.

Lemma 4: Suppose that Assumptions 1 and 3 hold. If $\Delta_0 < N$, then

$$\begin{aligned}&E\big\{V_N^*(\hat{x}(t_1|t_1^-))-V_N^*(\hat{x}(t_0|t_0^-))\big|\xi(t_0),\mathcal{E}_1^0,\mathcal{E}_2^0,\mathcal{E}_1^1\big\}\\ \le&\Psi_0 E\{\|w\|^s\}-\sum_{i=0}^{\Delta_0-1} l(\hat{x}_i^*(t_0|t_0^-),u_i^*(t_0))\end{aligned}$$

where $\Psi_0=\lambda_f\beta_{0,N-\Delta_0}+\lambda_l\sum_{i=0}^{N-\Delta_0-1}\beta_{0,i}$.

Proof: For the proof, we follow similar line presented in [13]. At time t_0, we use the calculated optimal control sequence $\mathbf{u}^*(t_0)$, and at time t_1, we construct a feasible control sequence $\{u_{\Delta_0}^*(t_0),\ldots,u_{N-1}^*(t_0),u_N^{\#},\ldots,u_{N+\Delta_0-1}^{\#}\}$, where $u_i^{\#}=\kappa(\hat{x}_{i-\Delta_0}(t_1|t_1^-)), i=N,\ldots,N+\Delta_0-1$.

The result can be obtained by calculating $V_N(\hat{x}(t_1|t_1^-))-V_N^*(\hat{x}(t_0|t_0^-))$, which is omitted here. ∎

Lemma 5: Suppose that Assumption 3 and 4 hold. If $\Delta_0 > N$, then there exists positive constant Ω_0 such that

$$\begin{aligned}&E\big\{V_N^*(\hat{x}(t_1|t_1^-))-V_N^*(\hat{x}(t_0|t_0^-))\big|\xi(t_0),\mathcal{E}_1^0,\mathcal{E}_2^0,\mathcal{E}_1^1\big\}\\ \le&(\gamma^{\Delta_0-N}-1)V_f(\hat{x}_N^*(t_0|t_0^-))\\ &-\sum_{i=0}^{N-1} l(\hat{x}_i^*(t_0|t_0^-),u_i^*(t_0))+\Omega_0 E\{\|w\|^s\}\end{aligned}$$

Proof: Given the conditions that $\mathcal{E}_1^0,\mathcal{E}_2^0,\mathcal{E}_1^1$, the result for two cases, $t_1^-\le t_0+N\le t_1$ and $t_0+N<t_1^-\le t_1$, is obtained by virtue of Lemma 1 and Assumption 4. ∎

In what follows, we can come to a preliminary conclusion according to the above results.

Theorem 1: Suppose that Assumption 1, 3, 4 hold, $p_a p_s\lambda_x<1$, and the prediction horizon N is chosen such that

$$\Big[\frac{p_a^{N+1}(1-p_s)}{1-p_a\gamma}-\frac{p_s^{N+1}(1-p_a)}{1-p_s\gamma}\Big]\frac{\gamma-1}{p_a-p_s}<\frac{\mu}{1-\mu}$$

for $p_a\neq p_s$, and

$$\frac{p_s^N\big[N(1-p_s)(1-p_s\gamma)+(1-p_s^2\gamma)\big]}{(1-p_s\gamma)^2/(\gamma-1)}<\frac{\mu}{1-\mu}$$

for $p_a=p_s$, where $\mu\triangleq\inf\frac{l(x)}{V_f(x)}$, then there exist constants C_1 and ρ such that

$$\begin{aligned}&E\{J(x(t_1))|\xi(t_0)\}\\ \le&C_1E\{\|w\|^s\}+(1-\rho)E\{J(x(t_0))|\xi(t_0)\}\end{aligned}\tag{18}$$

Proof: We only deal with the $p_a\neq p_s$ case. For $p_a=p_s$, the result can be obtained by similar method. Due to the limited space, a short sketch of the proof is presented in the following. Notice that

$$\begin{aligned}&J(x(t_1))-J(x(t_0))\\ \le&\big|J(x(t_1))-V_N^*(\hat{x}(t_1|t_1^-))\big|+V_N^*(\hat{x}(t_1|t_1^-))\\ &-V_N^*(\hat{x}(t_0|t_0^-))+\big|V_N^*(\hat{x}(t_0|t_0^-))-J(x(t_0))\big|\end{aligned}$$

First, we obtain that

$$\begin{aligned}&E\{|J(x(t_1))-V_N^*(\hat{x}(t_1|t_1^-))|\big|\xi(t_0),\mathcal{E}_1^0\}\\ \le&\bar{\lambda}_N\sum_{q_1=0}^{\infty}\frac{1-\lambda_x^{q_1}}{1-\lambda_x}[(p_sp_a)^{q_1}-(p_sp_a)^{q_1+1}]E\{\|w\|^s\}\end{aligned}$$

and

$$\begin{aligned}&E\{|J(x(t_0))-V_N^*(\hat{x}(t_0|t_0^-))|\big|\xi(t_0),\mathcal{E}_1^0\}\\ \le&\bar{\lambda}_N\frac{1-\lambda_x^{q_0}}{1-\lambda_x}E\{\|w\|^s\}\end{aligned}$$

By the total probability formula and some rearrangements, it yields

$$\begin{aligned}&E\{V_N^*(\hat{x}(t_1|t_1^-))-V_N^*(\hat{x}(t_0|t_0^-))\big|\xi(t_0),\mathcal{E}_1^0\}\\ \le&\Gamma_{q_0}E\{\|w\|^s\}+\Lambda_N V_f(\hat{x}_N^*(t_0|t_0^-))-l(\hat{x}^*(t_0|t_0^-),u_0^*(t_0))\end{aligned}$$

where $\Lambda_N=\big[\frac{p_a^{N+1}(1-p_s)}{1-p_a\gamma}-\frac{p_s^{N+1}(1-p_a)}{1-p_s\gamma}\big]\frac{\gamma-1}{p_a-p_s}$ and Γ_{q_0} are two finite constants.

Since $V_f(\hat{x}_N^*(t_0|t_0^-))\le V_f(\hat{x}^*(t_0|t_0^-))$, and N is chosen such that $\big[\frac{p_a^{N+1}(1-p_s)}{1-p_a\gamma}-\frac{p_s^{N+1}(1-p_a)}{1-p_s\gamma}\big]\frac{\gamma-1}{p_a-p_s}<\frac{\mu}{1-\mu}$, where $\mu\triangleq\inf\frac{l(x)}{V_f(x)}$, we have $l(\hat{x}^*(t_0|t_0^-))-\Lambda_N V_f(\hat{x}_N^*(t_0|t_0^-))\ge\rho V_f(\hat{x}^*(t_0|t_0^-))\ge\rho V_N^*(\hat{x}^*(t_0|t_0^-))$, where $\rho\triangleq-(\Lambda_N-(1+\Lambda_N)\mu)>0$. Then, according to the definition of the conditional expectation, we have

$$\begin{aligned}&E\{J(x(t_1))-J(x(t_0))|\xi(t_0)\}\\ =&\sum_{q_0=0}^{\infty}E\{J(x(t_1))-J(x(t_0))|\xi(t_0),\mathcal{E}_1^0\}\Pr\{\mathcal{E}_1^0\}\\ \le&C_1E\{\|w\|^s\}-\rho E\{J(x(t_0))|\xi(t_0)\}\end{aligned}$$

If $p_sp_a\lambda_x<1$ holds, it can be verified that C_1 is finite. ∎

Remark 5: In Theorem 1, the relation of prediction horizon and two packet loss probabilities is derived, that is $\big[\frac{p_a^{N+1}(1-p_s)}{1-p_a\gamma}-\frac{p_s^{N+1}(1-p_a)}{1-p_s\gamma}\big]\frac{\gamma-1}{p_a-p_s}<\frac{\mu}{1-\mu}$. It can be observed that if $p_s=0$ (or $p_a=0$), the inequality becomes $\frac{p_a^N(\gamma-1)}{1-p_a\gamma}<\frac{\mu}{1-\mu}$. This is consistent with the results obtained in [16] and in [13] by letting $q=1-p$. So these results can be viewed as special cases of our result.

Theorem 2: Suppose that Assumptions 1 $\sim$ 4 hold, $p_sp_a\lambda_x<1$, and the prediction horizon N is chosen properly, then there exist constants D_1 and D_2 such that, for all $k\in\mathbb{N}_0$

$$\begin{aligned}&\max_{\tau\in\{t_k,t_k+1,\ldots,t_k+\Delta_k-1\}}E\{\|x(\tau)\|^s\}\\ \le&D_1(1-\rho)^kE\{\|x_0\|^s\}+D_2E\{\|w\|^s\}\end{aligned}\tag{19}$$

and $E\{\|x(\tau)\|^s\}\le D_1E\{\|x_0\|^s\}+D_2E\{\|w\|^s\}$ for all $\tau\in[t_{ini},t_0-1]$.

The proof is omitted here due to space limitations.

Remark 6: The above theorem indicates that if the conditions of the theorem are satisfied, then the closed loop system is stochastic stable and $E\{\|x(k)\|^s\}$ is bounded for all $k\ge t_{ini}$. Furthermore, one obtains that $\lim_{k\to\infty}E\{\|x(k)\|^s\}\le D_2E\{\|w\|^s\}$. Therefore, for a disturbance-free system, it yields $\lim_{k\to\infty}E\{\|x(k)\|^s\}=0$. Unlike the result stated in [17] where the input-to-state practical stability condition is established and the ultimate bound will not converge to zero even for the disturbance-free system.

V. Numerical Example

In this section, we show the effectiveness of our results by a numerical example.

Example 1: [13] We consider the following nonlinear system

$$x_1(k+1) = x_2(k) + u_1(k) + w_1(k)$$
$$x_2(k+1) = -sat(x_1(k) + x_2(k)) + u_2(k) + w_2(k)$$

where

$$sat(x) = \begin{cases} -1 & \text{if } x < -1, \\ x & \text{if } -1 \leq x \leq 1, \\ 1 & \text{if } x > 1. \end{cases}$$

and $w_1(k), w_2(k)$ are i.i.d Gaussian distribution $N(0, 0.01)$. The constraint of the second component of control input is $u_2(k) \in [-0.8, 0.8], \forall k \in \mathbb{N}_0$. As analyzed in [13], we have $\lambda_x = 1.618$, $\lambda_w = 1$ and $s = 1$. If we choose the weighting function as $l(x, u) = \|x\| + 0.5\|u\|$ and $V_f(x) = 2\|x\|$, then we obtain the parameters $\lambda_l = \alpha_l = 1$, $\lambda_f = \alpha_f = 2$ such that Assumption 1 and Assumption 2 are satisfied. The constrained control law κ is chosen as $\kappa(x) = [-x_2 \quad 0.8sat(x_1 + x_2)]^T$, Assumption 3 is then satisfied. Besides, the introduced network brings the random packet losses. Suppose $p_s = 0.5$ and $p_a = 0.3$, then we have $p_s p_a \lambda_x < 1$. When $\gamma = 1.618$ and $\eta = 2$, the open-loop system satisfies Assumption 4 with $p_a\gamma < 1$ and $p_s\gamma < 1$. To guarantee the stability, the prediction horizon based on Theorem 1 should satisfy $N \geq 3$ with $\mu = 0.5$. We choose $N = 3$ here.

Fig. 3 illustrates the state response and the possible realizations of the random packet losses of two channels (0 means packet loss occurs). For C-A channel, there are no values at some time instants. This implies the smart controller does not take any action at these time instants. From the first subfigure, we can conclude that the state will converge ultimately to a bounded set.

VI. Conclusion

We have studied the NCS where the packet-based model predictive controller communicates with sensor and actuator through two unreliable networks suffered from stochastic packet losses. To establish the stability conditions, a new cost function, which depends on the actual state and control sequence calculated by estimated state, has been constructed. Furthermore, we have given the relation of prediction horizon and packet loss probability to guarantee the stochastic stability. Finally, the effectiveness has been verified by a numerical example. Future research could include the study of MPC-based scheduling problem, including decentralized scheduling [18] and centralized scheduling [19].

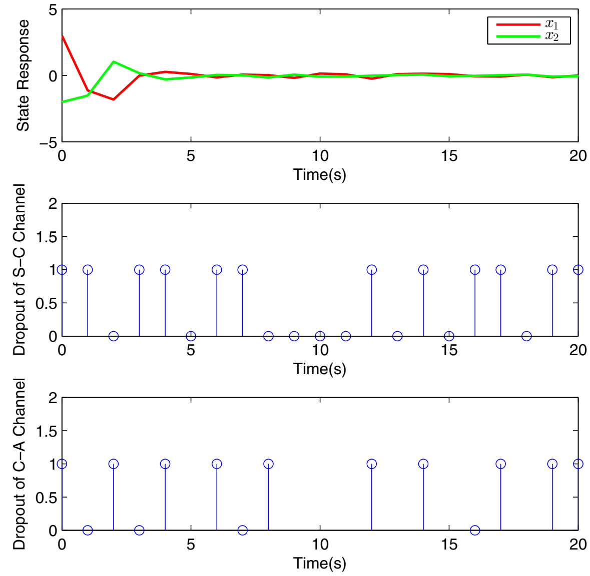

Fig. 3. The state response and packet losses of two channels

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